

Derivations and automorphisms of twisted deformed Schrödinger-Virasoro Lie algebras

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Abstract: In this paper the derivation algebra and automorphism group of the twisted deformed Schrödinger-Virasoro Lie algebras are determined.

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1 Introduction

It is well known that the Virasoro algebra plays an important role in many areas of theoretical physics and mathematics, which occurs in the investigation of conformal field theory and has a \mathbb{C} -basis $\{L_n, c \mid n \in \mathbb{Z}\}$ with the nontrivial relations $[L_n, L_m] = (m-n)L_{n+m}$. It can be regarded as the complexification of the Lie algebra of polynomial vector fields on a circle, and also as the Lie algebra of derivations of the ring $\mathbb{C}[z, z^{-1}]$. The centerless Virasoro algebra admits many kinds of extensions, one of these is the Schrödinger-Virasoro type Lie algebras (see [7–9]), firstly introduced in [2] in the context of non-equilibrium statistical physics during the process of investigating the free Schrödinger equations and closely related to the Schrödinger algebra and the Virasoro algebra. Recently the vertex algebra representations, Lie bialgebra structures, irreducible weight modules with finite-dimensional weight spaces and also Wittaker modules of the Schrödinger-Witt Lie algebras were extensively investigated in [10–12, 16]. The generalization of the Schrödinger-Virasoro Lie algebras was introduced in [15], whose automorphism groups and Verma modules were described therein.

For any $\lambda, \mu \in \mathbb{C}$, [9] introduced a family of infinite-dimensional Lie algebras called *twisted deformed Schrödinger-Virasoro Lie algebras*, admitting \mathbb{C} -basis $\{L_n, Y_n, M_n \mid n \in \mathbb{Z}\}$ and the following Lie brackets

$$\begin{aligned} [L_n, L_m] &= (m-n)L_{n+m}, \\ [L_n, Y_m] &= (m - \frac{\lambda+1}{2}n + \mu)Y_{n+m}, \quad [Y_n, Y_m] = (m-n)M_{n+m}, \\ [L_n, M_m] &= (m - \lambda n + 2\mu)M_{n+m}, \quad [Y_n, M_m] = [M_n, M_m] = 0. \end{aligned}$$

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We denote this Lie algebra by \mathcal{L} , which is \mathbb{Z} -graded with

$$\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \mathcal{L}_n, \quad \mathcal{L}_n = \mathbb{C} L_n \oplus \mathbb{C} Y_n \oplus \mathbb{C} M_n, \quad [\mathcal{L}_n, \mathcal{L}_m] \subseteq \mathcal{L}_{n+m}.$$

For convenience we introduce the following notations

$$\mathcal{L} = \sum_{n \in \mathbb{Z}} \mathbb{C} L_n, \quad \mathcal{Y} = \sum_{n \in \mathbb{Z}} \mathbb{C} Y_n, \quad \mathcal{M} = \sum_{n \in \mathbb{Z}} \mathbb{C} M_n.$$

Then \mathcal{L} is the centerless Virasoro algebra, $\mathcal{I} = \mathcal{Y} \oplus \mathcal{M}$ is the unique maximal ideal of \mathcal{L} and \mathcal{M} is the center of \mathcal{L} .

It is well known that the determination of derivations and automorphisms plays an important part in the investigation of the structure and representation of the relevant Lie algebras. Many references (see [1, 3–6, 9, 17, 18]) have focused on derivations and automorphisms of different Lie algebra backgrounds.

Note that if $\mu \in \mathbb{Z}$, then $\{L_n, Y_{n-\mu}, M_{n-2\mu}\}$ is a basis of \mathcal{L} . Hence one can assume $\mu = 0$ if $\mu \in \mathbb{Z}$. Especially, for the case $\lambda = \mu = 0$, \mathcal{L} is nothing but the twisted Schrödinger-Virasoro algebra, whose derivations and automorphisms were determined in [13]. If $\mu \in \frac{1}{2} + \mathbb{Z}$, one can assume $\mu = \frac{1}{2}$ by shifting basis, in which case the derivations were investigated in [14] and automorphisms for the special case $\lambda = 0$ were determined in [15]. Furthermore, for the case $\mu = \frac{1}{2}$ and $\lambda \neq 0$, one can obtain the corresponding results on automorphisms by following the proof of Theorem 3.2 in [15]. Thus, in this paper we always make the following assumptions on μ and λ

$$\mu \notin \frac{1}{2} + \mathbb{Z}, \quad \mu = 0 \quad \text{and} \quad \lambda \neq 0 \quad \text{if} \quad \mu \in \mathbb{Z}. \quad (1.1)$$

In the following two sections, we shall determine the derivation algebra (see Theorem 2.5) and the automorphism group (see Theorem 3.4) of \mathcal{L} under the assumptions made in (1.1).

2 Derivation algebra of \mathcal{L}

A linear map $d : \mathcal{L} \rightarrow \mathcal{L}$ is called a *derivation* of \mathcal{L} , if $d([x, y]) = [d(x), y] + [x, d(y)]$ holds for any $x, y \in \mathcal{L}$. For any fixed $z \in \mathcal{L}$, the linear map $ad_z : \mathcal{L} \rightarrow \mathcal{L}$ is called an *inner* derivation if $ad_z(x) = [z, x]$ for any $x \in \mathcal{L}$. Denote by $Der \mathcal{L}$ and $ad \mathcal{L}$ respectively the vector spaces of all derivations and inner derivations. Then the first homological group $\mathcal{H}^1(\mathcal{L}, \mathcal{L}) \cong Der \mathcal{L} / ad \mathcal{L}$.

Firstly, we give the following description of $Der \mathcal{L}$.

Lemma 2.1 $Der \mathcal{L} = (Der \mathcal{L})_0 + ad \mathcal{L}$, where

$$(Der \mathcal{L})_0 = \{d \in Der \mathcal{L} \mid d(\mathcal{L}_n) \subseteq \mathcal{L}_n, \forall n \in \mathbb{Z}\}.$$

Proof By Proposition 1.2 of [1], we need two steps to complete the proof of this lemma.

Step 1 For any $n \neq 0$, if $d \in \mathcal{H}^1(\mathcal{L}_0, \mathcal{L}_n)$, then $d \in \text{ad } \mathcal{L}$.

Since $\mathcal{L}_n = \mathbb{C}L_n \oplus \mathbb{C}Y_n \oplus \mathbb{C}M_n$ for all $n \in \mathbb{Z}$, one can assume $d(X_0) = e_1^X L_n + e_2^X Y_n + e_3^X M_n$ for some $e_i^X \in \mathbb{C}$, $i = 1, 2, 3$, $X \in \{L, Y, M\}$. Applying d to $[L_0, Y_0] = \mu Y_0$, $[L_0, M_0] = 2\mu M_0$ and $[Y_0, M_0] = 0$ respectively, we obtain

$$\begin{cases} (n - \mu)e_1^Y = (\mu - \frac{1+\lambda}{2}n)e_1^L + ne_2^Y = ne_2^L - (n + \mu)e_3^Y = 0, \\ (n - 2\mu)e_1^M = (n - \mu)e_2^M = (2\mu - \lambda n)e_1^L - ne_3^M = 0, \\ (2\mu - \lambda n)e_1^Y + ne_2^M = (\mu - \frac{1+\lambda}{2}n)e_1^M = 0, \end{cases}$$

which combined with (1.1) give

$$e_1^Y = e_1^M = e_2^M = 0, \quad e_2^Y = \frac{\lambda - 2\mu + 1}{2n}e_1^L, \quad e_3^Y = \frac{n}{n + \mu}e_2^L, \quad e_3^M = \frac{2\mu - \lambda n}{n}e_1^L.$$

Hence, denoting $\alpha = \frac{1}{n}e_1^L L_n + \frac{1}{n+\mu}e_2^L Y_n + \frac{1}{n+2\mu}e_3^L M_n$, we obtain $d(X_0) = [X_0, \alpha]$ for any $X \in \{L, Y, M\}$.

Step 2 For any $n \neq m$, if $f \in \text{Hom}_{\mathcal{L}_0}(\mathcal{L}_n, \mathcal{L}_m)$, then $f = 0$.

Assume that $f(X_n) = c_1^X L_m + c_2^X Y_m + c_3^X M_m$ for some $c_i^X \in \mathbb{C}$ with $X \in \{L, Y, M\}$, $i = 1, 2, 3$. Applying f to both sides of the following three identities

$$[L_0, L_n] = nL_n, \quad [L_0, Y_n] = (n + \mu)Y_n, \quad [L_0, M_n] = (n + 2\mu)M_n,$$

and comparing the coefficients of L_m , Y_m and M_m respectively, we have

$$\begin{cases} (m - n)c_1^L = (m - n + \mu)c_2^L = (m - n + 2\mu)c_3^L = 0, \\ (m - n - \mu)c_1^Y = (m - n)c_2^Y = (m - n + \mu)c_3^Y = 0, \\ (m - n - 2\mu)c_1^M = (m - n - \mu)c_2^M = (n + 2\mu)c_3^M = 0. \end{cases}$$

Note that μ satisfies (1.1), one can deduce $c_i^X = 0$, $X \in \{L, Y, M\}$, $i = 1, 2, 3$. Thus $f = 0$. \square

Let $d \in (\text{Der } \mathcal{L})_0$. For $n \in \mathbb{Z}$ and $X \in \{L, Y, M\}$, one can assume

$$d(X_n) = f_1^X(n)L_n + f_2^X(n)Y_n + f_3^X(n)M_n \quad \text{for some } f_i^X(n) \in \mathbb{C}, \quad i = 1, 2, 3.$$

Lemma 2.2 For any $n \in \mathbb{Z}$ and some $a, b, c, \bar{c}, e, \bar{e} \in \mathbb{C}$, one can assume

- (1) $f_1^L(n) = an$.
- (2) $f_2^L(n) = \begin{cases} \frac{e}{2\mu}(2\mu - (\lambda + 1)n) & \text{if } \mu \notin \mathbb{Z}, \\ bn(1 - \delta_{\lambda, -1}) & \text{if } \mu = 0. \end{cases}$

(3) If $\mu \notin \mathbb{Z}$, then $f_3^L(n) = \frac{\bar{c}}{2\mu}(2\mu - \lambda n)$.

(4) If $\mu = 0$, then

$$f_3^L(n) = \begin{cases} \frac{c}{6}(n^3 - n) - \frac{\bar{c}}{3}(n^3 - 4n) & \text{if } \lambda = -2, \\ \frac{c}{2}(n^2 - n) - \bar{c}(n^2 - 2n) & \text{if } \lambda = -1, \\ \bar{c}n & \text{if } \lambda \notin \{-2, 0, -1\}. \end{cases}$$

Proof (1) Applying d to $[L_n, L_m] = (m - n)L_{m+n}$ and comparing the coefficients of L_{m+n} , we have $(m - n)(f_1^L(m + n) - f_1^L(m) - f_1^L(n)) = 0$, which gives

$$f_1^L(m + n) = f_1^L(m) + f_1^L(n) \quad \text{if } m \neq n. \quad (2.1)$$

Taking $n = 0$ in (2.1), we have $f_1^L(0) = 0$. Applying (2.1), one can deduce

$$f_1^L(2n) = f_1^L(3n) + f_1^L(-n) = f_1^L(2n) + f_1^L(n) + f_1^L(-2n) + f_1^L(n) = 2f_1^L(n),$$

which together with (2.1) gives $f_1^L(m + n) = f_1^L(m) + f_1^L(n)$ for all $m, n \in \mathbb{Z}$. Thus $f_1^L(n) = f_1^L(n - 1) + f_1^L(1)$. Hence, by induction on n , one can deduce $f_1^L(n) = nf_1^L(1)$ for all $n \in \mathbb{Z}$.

(2) Applying d to $[L_n, L_m] = (m - n)L_{m+n}$ and comparing the coefficients of Y_{m+n} , one has

$$(2m - (\lambda + 1)n + 2\mu)f_2^L(m) - (2n - (\lambda + 1)m + 2\mu)f_2^L(n) = 2(m - n)f_2^L(m + n). \quad (2.2)$$

Case 1 $\mu \notin \mathbb{Z}$.

Setting $n = 0$ in (2.2), we obtain $2\mu f_2^L(m) = (2\mu - (\lambda + 1)m)f_2^L(0)$, which gives $f_2^L(m) = \frac{1}{2\mu}(2\mu - (\lambda + 1)m)f_2^L(0)$ for all $m \in \mathbb{Z}$.

Case 2 $\mu = 0$.

Taking $\mu = 0$ in (2.2), we have

$$(2m - (\lambda + 1)n)f_2^L(m) - (2n - (\lambda + 1)m)f_2^L(n) = 2(m - n)f_2^L(m + n). \quad (2.3)$$

Taking $n = 1, 2$ in (2.3), we obtain

$$2(m - 1)f_2^L(m + 1) = (2m - \lambda - 1)f_2^L(m) - (2 - (\lambda + 1)m)f_2^L(1), \quad (2.4)$$

$$2(m - 2)f_2^L(m + 2) = 2(m - \lambda - 1)f_2^L(m) - (4 - (\lambda + 1)m)f_2^L(2). \quad (2.5)$$

Taking $n = 1$ and replacing m by $m + 1$ in (2.3), one can deduce

$$(2m - \lambda + 1)f_2^L(m + 1) - (2 - (\lambda + 1)(m + 1))f_2^L(1) = 2mf_2^L(m + 2). \quad (2.6)$$

Multiplying (2.6) by $2(m-1)(m-2)$, then using (2.4) and (2.5), we obtain

$$\begin{aligned} & (\lambda-1)(2\lambda-(5+\lambda)m+2)f_2^L(m) \\ &= m(m-2)(4(1+\lambda)m-\lambda^2-7)f_2^L(1)+2m(m-1)(4-(\lambda+1)m)f_2^L(2). \end{aligned} \quad (2.7)$$

Applying d to $[L_n, Y_m] = (m - \frac{\lambda+1}{2}n)Y_{m+n}$ and comparing the coefficients of M_{m+n} , we have

$$2(m-n)f_2^L(n)+2(m-\lambda n)f_3^Y(m)-(2m-(1+\lambda)n)f_3^Y(n+m)=0. \quad (2.8)$$

Setting $m=0$ in (2.8), we obtain

$$(1+\lambda)nf_3^Y(n)=2nf_2^L(n)+2\lambda nf_3^Y(0). \quad (2.9)$$

Replacing n by $-n$, m by n in (2.8), one has

$$(\lambda+3)nf_3^Y(0)=2(1+\lambda)nf_3^Y(n)+4nf_2^L(-n). \quad (2.10)$$

Multiplying (2.9) by $(\lambda+3)$ and using (2.10), we obtain

$$3(1-\lambda^2)nf_3^Y(n)=2(\lambda+3)nf_2^L(n)+8\lambda nf_2^L(-n). \quad (2.11)$$

Taking $n=1, 2$ in (2.11) respectively, one has

$$3(1-\lambda^2)f_3^Y(1)=2(\lambda+3)f_2^L(1)+8\lambda f_2^L(-1), \quad (2.12)$$

$$3(1-\lambda^2)f_3^Y(2)=2(\lambda+3)f_2^L(2)+8\lambda f_2^L(-2). \quad (2.13)$$

Furthermore, setting $n=m=1$ in (2.8), we obtain

$$(1-\lambda)(f_3^Y(2)-2f_3^Y(1))=0. \quad (2.14)$$

Multiplying (2.14) by $3(1+\lambda)$, using (2.12) and (2.13), one can deduce

$$(\lambda+3)(f_2^L(2)-2f_2^L(1))+4\lambda(f_2^L(-2)-2f_2^L(-1))=0. \quad (2.15)$$

Subcase 1 $\lambda=-5$.

Taking $\lambda=-5$ in (2.7), one has

$$f_2^L(m)=\frac{1}{6}m(m-1)(m+1)f_2^L(2)-\frac{1}{3}m(m-2)(m+2)f_2^L(1). \quad (2.16)$$

Taking $m=-1, -2$ in (2.16), we obtain $f_2^L(-1)=-f_2^L(1)$ and $f_2^L(-2)=-f_2^L(2)$. Then (2.15) gives $f_2^L(2)=2f_2^L(1)$, which together with (2.16), admits $f_2^L(m)=mf_2^L(1)$ for all $m \in \mathbb{Z}$.

Subcase 2 $\lambda = -3$.

Taking $\lambda = -3$ in (2.7), one has

$$2(m+2)f_2^L(m) = m(m+2)(m-1)f_2^L(2) - 2m(m+2)(m-2)f_2^L(1), \quad (2.17)$$

which gives

$$f_2^L(m) = \frac{1}{2}m(m-1)f_2^L(2) - m(m-2)f_2^L(1) \quad \text{if } m \neq -2. \quad (2.18)$$

Setting $\lambda = -3$, $m = 1$ and $n = -2$ in (2.3), we have

$$f_2^L(-2) = f_2^L(1) + 3f_2^L(-1). \quad (2.19)$$

Furthermore, taking $m = -1$ in (2.18), one has

$$f_2^L(-1) = f_2^L(2) - 3f_2^L(1). \quad (2.20)$$

Applying this by substituting for the second term of the right-hand side in (2.19), we have

$$f_2^L(-2) = 3f_2^L(2) - 8f_2^L(1). \quad (2.21)$$

Thus (2.18) holds for all $m \in \mathbb{Z}$. Taking $\lambda = -3$ in (2.15), then using (2.20) and (2.21), one can deduce $f_2^L(2) = 2f_2^L(1)$. Thus (2.18) gives $f_2^L(m) = mf_2^L(1)$ for all $m \in \mathbb{Z}$.

Subcase 3 $\lambda = -1$.

Letting $\lambda = -1$ in (2.7) gives

$$f_2^L(m) = (m-1)f_2^L(2) - (m-2)f_2^L(1) \quad \text{if } m \neq 0. \quad (2.22)$$

Taking $m = -1$ in (2.22), we have

$$f_2^L(-1) = -2f_2^L(2) + 3f_2^L(1). \quad (2.23)$$

Furthermore, letting $\lambda = -1$, $m = 1$, $n = -1$ in (2.3), we have $f_2^L(1) + f_2^L(-1) = 2f_2^L(0)$.

Using this in (2.23), one has $f_2^L(0) = -f_2^L(2) + 2f_2^L(1)$. Thus (2.22) holds for all $m \in \mathbb{Z}$.

Taking $m = -2$ in (2.22), we obtain

$$f_2^L(-2) = -3f_2^L(2) + 4f_2^L(1). \quad (2.24)$$

Taking $\lambda = -1$ in (2.15), then using (2.23) and (2.24), one can deduce $f_2^L(2) = 2f_2^L(1)$.

Thus (2.22) gives $f_2^L(m) = mf_2^L(1)$ for all $m \in \mathbb{Z}$. Taking $\lambda = -1$ in (2.12) and using $f_2^L(-1) = -f_2^L(1)$, one can deduce $f_2^L(1) = 0$. Thus $f_2^L(n) = 0$ for all $n \in \mathbb{Z}$.

Subcase 4 $\lambda = 1$.

Letting $\lambda = 1$ in (2.3), we obtain $(m - n)(f_2^L(m + n) - f_2^L(n) - f_2^L(m)) = 0$. Utilizing the similar technique to that of (1), one can deduce $f_2^L(m) = mf_2^L(1)$ for all $m \in \mathbb{Z}$.

Subcase 5 $\lambda \notin \{-5, -3, \pm 1\}$.

Taking $n = 0$ in (2.3), we have $(\lambda + 1)m f_2^L(0) = 0$ for all $m \in \mathbb{Z}$. This forces $f_2^L(0) = 0$. Using this and replacing n by $-m$ in (2.3), we obtain $(\lambda + 3)(f_2^L(m) + f_2^L(-m)) = 0$. Thus $f_2^L(m) = -f_2^L(-m)$ for all $m \in \mathbb{Z}$ since $\lambda \neq -3$. Thus

$$f_2^L(1) = -f_2^L(-1), \quad f_2^L(2) = -f_2^L(-2). \quad (2.25)$$

Taking $n = 2$ in (2.3), we have

$$2(m - \lambda - 1)f_2^L(m) - (4 - (\lambda + 1)m)f_2^L(2) = 2(m - 2)f_2^L(m + 2). \quad (2.26)$$

Replacing m by $m + 2$, n by -2 in (2.3) and using (2.25), we have

$$2(m + \lambda + 3)f_2^L(m + 2) - (4 + (\lambda + 1)(m + 2))f_2^L(2) = 2(m + 4)f_2^L(m). \quad (2.27)$$

Using (2.26) and (2.27), one can deduce $(\lambda + 5)(\lambda - 1)(2f_2^L(m) - mf_2^L(2)) = 0$, which gives $f_2^L(m) = \frac{1}{2}f_2^L(2)m$. Taking $m = 2$, $n = -1$ in (2.3) and using (2.25), we can deduce $f_2^L(2) = 2f_2^L(1)$. Thus $f_2^L(m) = mf_2^L(1)$ for all $m \in \mathbb{Z}$. We have completed the proof of (2).

Next we begin the proof of (3) and (4) of this lemma. Applying d to $[L_n, L_m] = (m - n)L_{m+n}$, then comparing the coefficients of M_{m+n} , one has

$$(m - \lambda n + 2\mu)f_3^L(m) - (n - \lambda m + 2\mu)f_3^L(n) = (m - n)f_3^L(m + n). \quad (2.28)$$

(3) If $\mu \notin \mathbb{Z}$, then taking $n = 0$ in (2.28), we have $2\mu f_3^L(m) = (2\mu - \lambda m)f_3^L(0)$, which gives

$$f_3^L(m) = \frac{1}{2\mu}(2\mu - \lambda m)f_3^L(0).$$

(4) Taking $\mu = 0$ in (2.28), we have

$$(m - \lambda n)f_3^L(m) - (n - \lambda m)f_3^L(n) = (m - n)f_3^L(m + n). \quad (2.29)$$

Taking $n = 1, 2$ in (2.29), we have

$$(m - \lambda)f_3^L(m) - (1 - \lambda m)f_3^L(1) = (m - 1)f_3^L(m + 1), \quad (2.30)$$

$$(m - 2\lambda)f_3^L(m) - (2 - \lambda m)f_3^L(2) = (m - 2)f_3^L(m + 2). \quad (2.31)$$

Setting $n = 1$ and replacing m by $m + 1$ in (2.29), one can deduce

$$(m + 1 - \lambda)f_3^L(m + 1) - (1 - \lambda(m + 1))f_3^L(1) = mf_3^L(m + 2). \quad (2.32)$$

Thus the three equations (2.30)–(2.32) imply

$$\begin{aligned} & (\lambda - 1)((\lambda + 2)m - 2\lambda)f_3^L(m) \\ &= m(m - 2)(2 - \lambda + \lambda^2 - 2\lambda m)f_3^L(1) - m(m - 1)(2 - \lambda m)f_3^L(2). \end{aligned} \quad (2.33)$$

Case 1 $\lambda = -2$.

Setting $\lambda = -2$ in (2.33), one has

$$f_3^L(m) = \frac{1}{6}m(m + 1)(m - 1)f_3^L(2) - \frac{1}{3}m(m - 2)(m + 2)f_3^L(1), \quad \forall m \in \mathbb{Z}.$$

Case 2 $\lambda = -1$.

In this case (2.33) gives

$$f_3^L(m) = \frac{1}{2}m(m - 1)f_3^L(2) - m(m - 2)f_3^L(1) \quad \text{for } m \neq -2. \quad (2.34)$$

Taking $\lambda = -1$, $m = -2$ and $n = 1$ in (2.29), one has

$$f_3^L(-2) = 3f_3^L(-1) + f_3^L(1). \quad (2.35)$$

Setting $m = -1$ in (2.34), we have $f_3^L(-1) = f_3^L(2) - 3f_3^L(1)$. Using this in (2.35), we obtain $f_3^L(-2) = 3f_3^L(2) - 8f_3^L(1)$. Thus (2.34) holds for all $m \in \mathbb{Z}$.

Case 3 $\lambda = 1$.

In this case (2.29) gives $(m - n)(f_3^L(m + n) - f_3^L(m) - f_3^L(n)) = 0$. Using the similar discussions to the proof of (1), we have $f_3^L(m) = mf_3^L(1)$ for all $m \in \mathbb{Z}$.

Case 4 $\lambda \notin \{-2, \pm 1, 0\}$.

Letting $n = 1$ in (2.29), we have

$$(m - \lambda)f_3^L(m) - (1 - \lambda m)f_3^L(1) = (m - 1)f_3^L(m + 1). \quad (2.36)$$

Replacing m by $m + 1$, n by -1 in (2.29), we obtain

$$(m + 1 + \lambda)f_3^L(m + 1) + (1 + \lambda(m + 1))f_3^L(-1) = (m + 2)f_3^L(m). \quad (2.37)$$

Thus, using (2.36) and (2.37), one can deduce

$$(1 - \lambda)(2 + \lambda)f_3^L(m) = (1 - \lambda m)(m + 1 + \lambda)f_3^L(1) - (m - 1)(\lambda(m + 1) + 1)f_3^L(-1). \quad (2.38)$$

Taking $m = -1$ in (2.38), we have $\lambda(1 + \lambda)(f_3^L(1) + f_3^L(-1)) = 0$, which forces $f_3^L(1) = -f_3^L(-1)$. Then (2.38) gives $(1 - \lambda)(\lambda + 2)(f_3^L(m) - mf_3^L(1)) = 0$. Thus $f_3^L(m) = mf_3^L(1)$ for all $m \in \mathbb{Z}$.

Hence denoted by $f_1^L(1) = a$, $f_2^L(1) = b$, $f_2^L(0) = e$, $f_3^L(0) = \bar{e}$, $f_3^L(2) = c$ and $f_3^L(1) = \bar{c}$, the lemma follows. \square

Lemma 2.3 *Define a , b and e as those given in Lemma 2.2. For any $n \in \mathbb{Z}$ and some \bar{a} , \hat{b} , $\bar{b} \in \mathbb{C}$, we have*

- (1) $f_1^Y(n) = 0$.
- (2) $f_2^Y(n) = an + \bar{a}$.
- (3) If $\mu \notin \mathbb{Z}$, then $f_3^Y(n) = -\frac{e}{\mu}n$.
- (4) If $\mu = 0$, then

$$f_3^Y(n) = \begin{cases} \hat{b}n & \text{if } \lambda = -1, \\ bn + \bar{b} & \text{if } \lambda = 1, \\ \frac{2b}{1+\lambda}n & \text{if } \lambda \neq \pm 1. \end{cases}$$

Proof (1) Applying d to $[L_n, Y_m] = (m - \frac{1+\lambda}{2}n + \mu)Y_{m+n}$ and comparing the coefficients of L_{m+n} , we have

$$2(m - n)f_1^Y(m) - (2m - (1 + \lambda)n + 2\mu)f_1^Y(n + m) = 0. \quad (2.39)$$

Case 1 $\mu \notin \mathbb{Z}$.

Setting $n = 0$ in (2.39), we have $\mu f_1^Y(m) = 0$, which gives $f_1^Y(m) = 0$ for all $m \in \mathbb{Z}$.

Case 2 $\mu = 0$.

In this case (2.39) gives

$$2(m - n)f_1^Y(m) - (2m - (1 + \lambda)n)f_1^Y(n + m) = 0. \quad (2.40)$$

Setting $m = 0$ in (2.40), we have

$$(1 + \lambda)nf_1^Y(n) = 2nf_1^Y(0). \quad (2.41)$$

Subcase 1 $\lambda \neq \pm 1$.

By (2.41), we have

$$nf_1^Y(n) = \frac{2}{1+\lambda}nf_1^Y(0), \quad \forall n \in \mathbb{Z}. \quad (2.42)$$

Multiplying (2.40) by $m(n+m)$ and using (2.42), we deduce $(\lambda-1)mn(m+n)f_1^Y(0) = 0$ for all $m, n \in \mathbb{Z}$, since $\lambda \neq 1$, which forces $f_1^Y(0) = 0$. Using this in (2.42), we obtain $f_1^Y(n) = 0$ for all $n \in \mathbb{Z}$.

Subcase 2 $\lambda = -1$.

Setting $\lambda = -1$ in (2.40), we have

$$(m-n)f_1^Y(m) - mf_1^Y(n+m) = 0. \quad (2.43)$$

Replacing n by $-m$ in (2.43), we have $f_1^Y(m) = \frac{1}{2}f_1^Y(0)$ for $m \neq 0$. Furthermore, letting $\lambda = -1$ in (2.41), we have $nf_1^Y(0) = 0$ for all $n \in \mathbb{Z}$, which forces $f_1^Y(0) = 0$. Thus $f_1^Y(m) = 0$ for all $m \in \mathbb{Z}$.

Subcase 3 $\lambda = 1$.

Letting $\lambda = 1$ in (2.40), we have

$$(m-n)(f_1^Y(m) - f_1^Y(n+m)) = 0. \quad (2.44)$$

Setting $m = 0$ in (2.44), we obtain $n(f_1^Y(n) - f_1^Y(0)) = 0$, which gives

$$f_1^Y(n) = f_1^Y(0), \quad \forall n \in \mathbb{Z}. \quad (2.45)$$

Applying d to $[Y_2, M_3] = 0$ and comparing the coefficients of M_5 , we have

$$f_2^M(3) = -f_1^Y(2) = -f_1^Y(0). \quad (2.46)$$

Furthermore, applying d to $[Y_1, Y_2] = M_3$ and comparing the coefficients, we have $f_2^M(3) = f_1^Y(1) + f_1^Y(2)$. This together with (2.45) and (2.46) gives $f_1^Y(0) = 0$. Thus $f_1^Y(n) = 0$ for all $n \in \mathbb{Z}$.

(2) From Lemma 2.2, we have $f_1^L(n) = an$ for some $a \in \mathbb{C}$. Thus applying d to $[L_n, Y_m] = (m - \frac{1+\lambda}{2}n + \mu)Y_{m+n}$ and comparing the coefficient of Y_{m+n} , we have

$$(2m - (1+\lambda)n + 2\mu)(f_2^Y(m) - f_2^Y(n+m) + an) = 0. \quad (2.47)$$

Case 1 $\lambda = -1$.

In this case (2.47) gives

$$(m + \mu)(f_2^Y(m) - f_2^Y(n + m) + an) = 0. \quad (2.48)$$

Recalling that μ satisfies (1.1), taking $m = 1$ and replacing n by $n - 1$ in (2.48), we obtain

$$f_2^Y(n) = a(n - 1) + f_2^Y(1), \quad \forall n \in \mathbb{Z}. \quad (2.49)$$

Taking $n = 0$ in (2.49), we obtain $f_2^Y(1) = f_2^Y(0) + a$. Thus (2.49) gives $f_2^Y(n) = an + f_2^Y(0)$ for all $n \in \mathbb{Z}$.

Case 2 $\lambda \neq -1$.

If $\mu = 0$, then setting $m = 0$ in (2.47), we obtain $(1 + \lambda)n(f_2^Y(0) - f_2^Y(n) + an) = 0$, which gives $f_2^Y(n) = an + f_2^Y(0)$ for all $n \in \mathbb{Z}$.

Suppose $\mu \notin \mathbb{Z}$. Replacing n by $-m$ in (2.47), we obtain

$$((3 + \lambda)m + 2\mu)(f_2^Y(m) - am - f_2^Y(0)) = 0. \quad (2.50)$$

If $(3 + \lambda)m + 2\mu \neq 0$ for all $m \in \mathbb{Z}$, then (2.50) gives $f_2^Y(m) = am + f_2^Y(0)$.

If $(3 + \lambda)m' + 2\mu = 0$ for some $m' \in \mathbb{Z}$, then replacing m by $m' + 1$ in (2.50), we have

$$(3 + \lambda)(f_2^Y(m' + 1) - a(m' + 1) - f_2^Y(0)) = 0. \quad (2.51)$$

It is obvious that $\lambda \neq -3$, otherwise $\mu = 0$. Thus (2.51) gives

$$f_2^Y(m' + 1) = a(m' + 1) + f_2^Y(0). \quad (2.52)$$

Choosing $n = 1$ and replacing m by m' in (2.47), we obtain

$$(2\mu - \lambda + 2m' - 1)(a + f_2^Y(m') - f_2^Y(m' + 1)) = 0. \quad (2.53)$$

Combining (2.52) and (2.53), one can deduce

$$(2\mu - \lambda + 2m' - 1)(f_2^Y(m') - am' - f_2^Y(0)) = 0. \quad (2.54)$$

Furthermore, using $2\mu = -(3 + \lambda)m'$ in (2.54), we have

$$(m' + 1)(1 + \lambda)(f_2^Y(m') - am' - f_2^Y(0)) = 0. \quad (2.55)$$

In this case $\lambda \neq -1$, we obtain

$$(m' + 1)(f_2^Y(m') - am' - f_2^Y(0)) = 0. \quad (2.56)$$

If $m' \neq -1$, then (2.56) gives $f_2^Y(m') = am' + f_2^Y(0)$.

Suppose that $m' = -1$. Then $2\mu = 3 + \lambda$. Setting $m = 0$, $n = -1$ in (2.47), we obtain

$$(\lambda + 2)(f_2^Y(0) - f_2^Y(-1) - a) = 0. \quad (2.57)$$

If $\lambda \neq -2$, then (2.57) forces $f_2^Y(-1) = -a + f_2^Y(0)$. If $\lambda = -2$, then setting $\lambda = -2$, $m = -1$ and $n = 1$ in (2.47), we obtain

$$(2\mu - 1)(a + f_2^Y(-1) - f_2^Y(0)) = 0, \quad (2.58)$$

since μ satisfies (1.1), which forces $f_2^Y(-1) = -a + f_2^Y(0)$. Hence $f_2^Y(m) = am + f_2^Y(0)$ for all $m \in \mathbb{Z}$. By now we have completed the proof of (2).

Next we begin the proof of (3) and (4) of this lemma. Applying d to $[L_n, Y_m] = (m - \frac{1+\lambda}{2}n + \mu)Y_{m+n}$ and comparing the coefficients of M_{m+n} , we have

$$2(m - n)f_2^L(n) + 2(m - \lambda n + 2\mu)f_3^Y(m) = (2m - (1 + \lambda)n + 2\mu)f_3^Y(n + m). \quad (2.59)$$

(3) If $\mu \notin \mathbb{Z}$, then taking $n = 0$ in (2.59), we obtain $f_3^Y(m) = -\frac{e}{\mu}m$ for $e = f_2^L(0)$.

(4) Taking $\mu = 0$ in (2.59), we obtain

$$2(m - n)f_2^L(n) + 2(m - \lambda n)f_3^Y(m) = (2m - (1 + \lambda)n)f_3^Y(n + m). \quad (2.60)$$

By Lemma 2.2 (2), one has $f_2^L(n) = bn(1 - \delta_{\lambda, -1})$ for some $b \in \mathbb{C}$.

Case 1 $\lambda = -1$.

In this case (2.60) gives

$$(m + n)f_3^Y(m) = mf_3^Y(m + n). \quad (2.61)$$

Taking $m = 1$ and replacing n by $n - 1$ in (2.61), we have $f_3^Y(n) = nf_3^Y(1)$ for all $n \in \mathbb{Z}$.

Case 2 $\lambda = 1$.

Letting $\lambda = 1$ in (2.60), we obtain

$$(m - n)(f_3^Y(n + m) - f_3^Y(m) - bn) = 0. \quad (2.62)$$

Thus (2.62) gives $f_3^Y(n) = bn + f_3^Y(0)$ for all $n \in \mathbb{Z}$.

Case 3 $\lambda \neq \pm 1$.

Setting $m = 0$ in (2.60), we have

$$f_3^Y(n) = \frac{2}{1 + \lambda}(bn + \lambda f_3^Y(0)) \quad \text{for } n \neq 0. \quad (2.63)$$

Furthermore, taking $n = \pm 1$ in (2.63) respectively, we have

$$f_3^Y(1) = \frac{2}{1+\lambda}(\lambda f_3^Y(0) + b), \quad f_3^Y(-1) = \frac{2}{1+\lambda}(\lambda f_3^Y(0) - b). \quad (2.64)$$

Taking $n = 1, m = -1$ in (2.60) and using (2.64), one can deduce $(\lambda - 1)f_3^Y(0) = 0$, which forces $f_3^Y(0) = 0$ since $\lambda \neq 1$. Then (2.63) implies $f_3^Y(n) = \frac{2b}{1+\lambda}n$ for all $n \in \mathbb{Z}$.

Hence, denoted by $f_2^Y(0) = \bar{a}$, $f_3^Y(1) = \hat{b}$ and $f_3^Y(0) = \bar{b}$, the lemma follows. \square

Lemma 2.4 *Let a and \bar{a} be as those given in Lemmas 2.2 and 2.3 respectively. For any $n \in \mathbb{Z}$, we have*

$$f_1^M(n) = f_2^M(n) = 0, \quad f_3^M(n) = an + 2\bar{a}.$$

Proof It follows from Lemmas 2.2 and 2.3 that there exists some a and \bar{a} such that

$$f_1^Y(n) = 0, \quad f_2^Y(n) = an + \bar{a}, \quad \forall n \in \mathbb{Z}.$$

Applying d to $[Y_n, Y_m] = (m - n)M_{m+n}$ and comparing the coefficients of L_{m+n} , Y_{m+n} and M_{m+n} respectively, we have

$$(m - n)f_1^M(m + n) = (m - n)f_2^M(m + n) = 0, \quad (2.65)$$

$$(m - n)(f_3^M(m + n) - a(n + m) - 2\bar{a}) = 0. \quad (2.66)$$

Setting $m = 0$ and replacing m by $-n$ in (2.65), we obtain

$$nf_1^M(n) = nf_2^M(n) = 0, \quad nf_1^M(0) = nf_2^M(0) = 0, \quad \forall n \in \mathbb{Z}.$$

This gives $f_1^M(n) = f_2^M(n) = 0$ for all $n \in \mathbb{Z}$.

Taking $m = 0$ in (2.66), one can deduce $f_3^M(n) = an + 2\bar{a}$ for $n \neq 0$. Taking $m = 1$ and $n = -1$ in (2.66), one can deduce $f_3^M(0) = 2\bar{a}$. Thus $f_3^M(n) = an + 2\bar{a}$ for all $n \in \mathbb{Z}$. \square

We construct some possible outer derivations of \mathcal{L} . Under the condition $\mu = 0$, for each $\lambda \in \{-2, \pm 1\}$, the following maps D_λ or \overline{D}_λ defined by

$$D_{-2}(L_n) = n^3 M_n, \quad D_{-1}(L_n) = n^2 M_n, \quad \overline{D}_{-1}(Y_n) = n M_n, \quad D_1(Y_n) = M_n,$$

are outer derivations of \mathcal{L} , where all other terms are vanishing. Besides, we can define another outer derivation D of \mathcal{L} , which does not depend on λ and μ :

$$D : L_n \mapsto 0, \quad Y_n \mapsto Y_n, \quad M_n \mapsto 2M_n.$$

It is easy to verify that for each $\lambda \in \{-2, \pm 1\}$, D , D_λ and \overline{D}_λ are linearly independent.

Theorem 2.5 (1) If $\mu \notin \frac{1}{2}\mathbb{Z}$, or $\mu = 0$ but $\lambda \notin \{-2, 0, \pm 1\}$, then $\text{Der}\mathcal{L} = \text{ad}\mathcal{L} \oplus \mathbb{C}D$.
(2) If $\mu = 0$, then

$$\text{Der}\mathcal{L} = \begin{cases} \text{ad}\mathcal{L} \oplus \mathbb{C}D \oplus \mathbb{C}D_{-2}, & \lambda = -2, \\ \text{ad}\mathcal{L} \oplus \mathbb{C}D \oplus \mathbb{C}D_{-1} \oplus \mathbb{C}\overline{D}_{-1}, & \lambda = -1, \\ \text{ad}\mathcal{L} \oplus \mathbb{C}D \oplus \mathbb{C}D_1, & \lambda = 1. \end{cases} \quad (2.67)$$

Proof Take $d_0 \in (\text{Der}\mathcal{L})_0$.

(1) Suppose $\mu \notin \frac{1}{2}\mathbb{Z}$. It follows from Lemmas 2.2–2.4 that there exist a, \bar{a}, e and $\bar{e} \in \mathbb{C}$ such that

$$\begin{aligned} d_0(L_n) &= anL_n + \frac{e}{2\mu}(2\mu - (\lambda + 1)n)Y_n + \frac{\bar{e}}{2\mu}(2\mu - \lambda n)M_n, \\ d_0(Y_n) &= (an + \bar{a})Y_n - \frac{e}{\mu}nM_n, \quad d_0(M_n) = (an + 2\bar{a})M_n. \end{aligned}$$

Set $\alpha = aL_0 - e\mu^{-1}Y_0 - \bar{e}(2\mu)^{-1}M_0$, then $d_0 = \text{ad}_\alpha + (\bar{a} - a\mu)D$.

If $\mu = 0$ and $\lambda \notin \{-2, 0, \pm 1\}$, by Lemmas 2.2–2.4, there exist a', \bar{a}', b' and $e' \in \mathbb{C}$ such that

$$\begin{aligned} d_0(L_n) &= a'nL_n + b'nY_n + e'nM_n, \\ d_0(Y_n) &= (a'n + \bar{a}')Y_n + \frac{2b'}{1 + \lambda}nM_n, \quad d_0(M_n) = (a'n + 2\bar{a}')M_n. \end{aligned}$$

Set $\beta = a'L_0 + 2b'(1 + \lambda)^{-1}Y_0 + e'\lambda^{-1}M_0$. Then $d_0 = \text{ad}_\beta + \bar{a}'D$.

(2) We shall divide the proof of (2.67) into the following four cases.

Case 1 $\lambda = -2$.

By Lemmas 2.2–2.4, there exist a_1, \bar{a}_1, b_1, c_1 and $\bar{c}_1 \in \mathbb{C}$ such that

$$\begin{aligned} d_0(L_n) &= a_1nL_n + b_1nY_n + \left(\frac{c_1}{6}(n^3 - n) - \frac{\bar{c}_1}{3}(n^3 - 4n)\right)M_n, \\ d_0(Y_n) &= (a_1n + \bar{a}_1)Y_n - 2b_1nM_n, \quad d_0(M_n) = (a_1n + 2\bar{a}_1)M_n. \end{aligned}$$

Set $\alpha_1 = a_1L_0 - 2b_1Y_0 + \frac{1}{12}(c_1 - 8\bar{c}_1)M_0$. Then we obtain $d_0 = \text{ad}_{\alpha_1} + \bar{a}_1D + \frac{1}{6}(c_1 - 2\bar{c}_1)D_{-2}$.

Case 2 $\lambda = -1$.

By Lemmas 2.2–2.4, there exist a_2, \bar{a}_2, b_2, c_2 and $\bar{c}_2 \in \mathbb{C}$ such that

$$\begin{aligned} d_0(L_n) &= a_2nL_n + \left(\frac{c_2}{2}(n^2 - n) - \bar{c}_2(n^2 - 2n)\right)M_n, \\ d_0(Y_n) &= (a_2n + \bar{a}_2)Y_n + b_2nM_n, \quad d_0(M_n) = (a_2n + 2\bar{a}_2)M_n. \end{aligned}$$

Set $\alpha_2 = a_2L_0 + \frac{1}{2}(c_2 - 4\bar{c}_2)M_0$. Then $d_0 = \text{ad}_{\alpha_2} + \bar{a}_2D + \frac{1}{2}(c_2 - 2\bar{c}_2)D_{-1} + b_2\overline{D}_{-1}$.

Case 3 $\lambda = 1$.

By Lemmas 2.2–2.4, there exist $a_4, \bar{a}_4, b_4, \bar{b}$ and $c_4 \in \mathbb{C}$ such that

$$\begin{aligned} d_0(L_n) &= a_4 n L_n + b_4 n Y_n + c_4 n M_n, \\ d_0(Y_n) &= (a_4 n + \bar{a}_4) Y_n + (b_4 n + \bar{b}) M_n, \quad d_0(M_n) = (a_4 n + 2\bar{a}_4) M_n. \end{aligned}$$

Set $\alpha_4 = a_4 L_0 + b_4 Y_0 + c_4 M_0$. Then $d_0 = ad_{\alpha_4} + \bar{a}_4 D + \bar{b} D_1$.

Hence, the theorem follows from Lemma 2.1. \square

3 Automorphism group of \mathcal{L}

In this section we denote by \mathbb{C}^* the set of all nonzero complex numbers and $Aut\mathcal{L}$ and $Inn\mathcal{L}$ the sets of automorphisms and inner automorphisms of \mathcal{L} . Firstly we need to introduce some technical lemmas.

Lemma 3.1 *Let $\sigma \in Aut\mathcal{L}$, $\epsilon \in \{\pm 1\}$ with $\epsilon = 1$ for $\mu \notin \frac{1}{2}\mathbb{Z}$. Then there exist some $\alpha, \beta \in \mathbb{C}^*$, $y_n \in \mathcal{Y}$ and $m_n, m'_n \in \mathcal{M}$ such that*

$$\begin{aligned} (1) \quad & \sigma(L_n) = \epsilon \alpha^n L_{\epsilon n} + y_n + m_n, \\ (2) \quad & \sigma(Y_n) = \alpha^n \beta Y_{\epsilon n} + m'_n, \\ (3) \quad & \sigma(M_n) = \epsilon \alpha^n \beta^2 M_{\epsilon n}. \end{aligned}$$

Proof (1) Note that $\sigma|_{\mathcal{L}}$ is an automorphism of the Witt algebra, so there exist some $\epsilon \in \{\pm 1\}$ and $\alpha \in \mathbb{C}^*$ such that

$$\sigma(L_n) = \epsilon \alpha^n L_{\epsilon n} + y_n + m_n \quad \text{for some } y_n \in \mathcal{Y}, m_n \in \mathcal{M}. \quad (3.1)$$

(2) Since $\mathcal{I} = \mathcal{Y} \oplus \mathcal{M}$ is the unique maximal ideal of \mathcal{L} , one can write

$$\sigma(Y_n) = \sum_{i \in S} b_{n_i} Y_i + m'_n, \quad (3.2)$$

for some $b_{n_i} \in \mathbb{C}^*$, $m'_n \in \mathcal{M}$ and $S \subseteq \mathbb{Z}$. By (3.1), there exist some $y_0 \in \mathcal{Y}$, $m_0 \in \mathcal{M}$ such that $\sigma(L_0) = \epsilon L_0 + y_0 + m_0$. Then applying σ to $(n + \mu)Y_n = [L_0, Y_n]$, we have

$$(n + \mu) \left(\sum_{i \in S} b_{n_i} Y_i + m'_n \right) = [\epsilon L_0 + y_0 + m_0, \sum_{i \in S} b_{n_i} Y_i + m'_n] = \sum_{i \in S} \epsilon b_{n_i} (i + \mu) Y_i + m''_n,$$

for some $m''_n \in \mathcal{M}$. Comparing the coefficients of Y_i , we obtain

$$b_{n_i} (i - \epsilon n - (\epsilon - 1)\mu) = 0, \quad \forall i \in S. \quad (3.3)$$

If $\mu = 0$, then (3.3) gives $i = \epsilon n$ for all $i \in S$. If $\mu \notin \frac{1}{2}\mathbb{Z}$, then (3.3) forces $\epsilon = 1$, in which case (3.3) gives $i = n$ for all $i \in S$. Hence, by (3.2), we can write

$$\sigma(Y_n) = b_n Y_{\epsilon n} + m'_n, \quad (3.4)$$

for some $b_n \in \mathbb{C}^*$ with $\epsilon = 1$ for $\mu \notin \frac{1}{2}\mathbb{Z}$.

Applying σ to $(m - \frac{\lambda+1}{2}n + \mu)Y_{m+n} = [L_n, Y_m]$ and using (3.1) and (3.4), we obtain

$$\begin{aligned} & (m - \frac{\lambda+1}{2}n + \mu)(b_{m+n}Y_{\epsilon(m+n)} + m'_{m+n}) \\ &= [\epsilon\alpha^n L_{\epsilon n} + y_n + m_n, b_m Y_{\epsilon m} + m'_m] = \alpha^n b_m (m - \frac{\lambda+1}{2}n + \mu)Y_{\epsilon(m+n)} + \bar{m}_n, \end{aligned}$$

for some $\bar{m}_n \in \mathcal{M}$. Thus we have

$$(m - \frac{\lambda+1}{2}n + \mu)((b_{m+n} - \alpha^n b_m)Y_{\epsilon(m+n)} + m'_{m+n}) - \bar{m}_n = 0,$$

which together with the fact $m'_{m+n}, \bar{m}_n \in \mathcal{M}$, gives

$$(2m - (\lambda+1)n + 2\mu)(b_{m+n} - \alpha^n b_m) = 0. \quad (3.5)$$

Taking $m = 0$ in (3.5), one has

$$(2\mu - (\lambda+1)n)(b_n - \alpha^n b_0) = 0. \quad (3.6)$$

If $2\mu - (\lambda+1)n' = 0$ for some $n' \in \mathbb{Z}$, then replacing n by $-n'$, m by n' in (3.5), we have

$$(\lambda+2)n'(b_{n'} - \alpha^{n'} b_0) = 0. \quad (3.7)$$

If $\lambda \neq -2$, then (3.7) gives $b_{n'} = \alpha^{n'} b_0$.

If $\lambda = -2$, then $2\mu + n' = 0$, since μ satisfies (1.1), which forces $n' = 0$. Thus

$$b_{n'} = \alpha^{n'} b_0 \quad \text{if } 2\mu - (\lambda+1)n' = 0. \quad (3.8)$$

Hence, (3.6) together with (3.8) gives $b_n = \alpha^n b_0$ for all $n \in \mathbb{Z}$. Using this in (3.4), one can write

$$\sigma(Y_n) = \alpha^n b_0 Y_{\epsilon n} + m'_n, \quad (3.9)$$

for some $m'_n \in \mathcal{M}$.

(3) Applying σ to $(n-2m)M_n = [Y_m, Y_{n-m}]$ and using (3.9), we have $(n-2m)\sigma(M_n) = [\alpha^m b_0 Y_{\epsilon m} + m'_m, \alpha^{n-m} b_0 Y_{\epsilon(n-m)} + m'_{n-m}]$ for some $m'_m, m'_{n-m} \in \mathcal{M}$, which gives

$$(n-2m)(\sigma(M_n) - \epsilon\alpha^n b_0^2 M_{\epsilon n}) = 0. \quad (3.10)$$

Letting $m = 0$ in (3.10), we have $\sigma(M_n) = \epsilon\alpha^n b_0^2 M_{\epsilon n}$ for $n \neq 0$. Furthermore, letting $n = 0$ in (3.10), one has $\sigma(M_0) = \epsilon b_0^2 M_0$. Thus we obtain $\sigma(M_n) = \epsilon\alpha^n b_0^2 M_{\epsilon n}$.

Hence, denoted by $b_0 = \beta$, the lemma follows. \square

Lemma 3.2 Let \bar{f} , f and g be \mathbb{C} -linear maps from \mathbb{Z} to \mathbb{C} . Define the \mathbb{C} -linear map $\phi : \mathcal{L} \rightarrow \mathcal{L}$ by

$$\phi(L_n) = L_n + \bar{f}(n)Y_n + f(n)M_n, \quad \phi(Y_n) = Y_n + g(n)M_n, \quad \phi(M_n) = M_n.$$

If $\mu = 0$ and $\phi \in \text{Aut}\mathcal{L}$, then there exist some $a, \bar{a}, b, c, \bar{c} \in \mathbb{C}$ such that

(1)

$$\bar{f}(n) = bn(1 - \delta_{\lambda, -1}), \quad g(n) = \begin{cases} an & \text{if } \lambda = -1, \\ bn + \bar{a} & \text{if } \lambda = 1, \\ \frac{2b}{1 + \lambda}n & \text{if } \lambda \neq \pm 1; \end{cases}$$

(2)

$$f(n) = \begin{cases} \frac{1}{6}c(n^3 - n) - \frac{1}{3}\bar{c}(n^3 - 4n) + \frac{1}{3}b^2n(n - 2)(n - 1) & \text{if } \lambda = -2, \\ \frac{1}{2}c(n^2 - n) - \bar{c}(n^2 - 2n) & \text{if } \lambda = -1, \\ \bar{c}n + \frac{b^2}{1 + \lambda}n(n - 1) & \text{if } \lambda \notin \{-2, 0, -1\}. \end{cases}$$

Proof Applying ϕ to $[L_n, L_m] = (m - n)L_{n+m}$ and comparing the coefficients of Y_{n+m} , M_{n+m} , we obtain

$$(2m - (1 + \lambda)n)\bar{f}(m) - (2n - (1 + \lambda)m)\bar{f}(n) = 2(m - n)\bar{f}(m + n), \quad (3.11)$$

$$(m - \lambda n)f(m) - (n - \lambda m)f(n) = (m - n)(f(m + n) - \bar{f}(n)\bar{f}(m)). \quad (3.12)$$

Applying ϕ to $[L_n, Y_m] = (m - \frac{\lambda+1}{2}n)Y_{n+m}$ and comparing the coefficients of M_{n+m} , we obtain

$$2(m - n)\bar{f}(n) + 2(m - \lambda n)g(m) = (2m - (\lambda + 1)n)g(n + m). \quad (3.13)$$

If we replace \bar{f} by f_2^L , g by f_3^Y in (3.11) and (3.13), then (3.11) and (3.13) become (2.3) and (2.60), respectively. Thus if we take $\bar{f}(1) = b$, $g(1) = a$ and $g(0) = \bar{a}$, then by Lemmas 2.2 and 2.3, (1) follows.

(2) If $\lambda = -1$, then noticing $\bar{f}(n) = 0$ in (3.12), we obtain

$$(m + n)f(m) - (n + m)f(n) = (m - n)f(m + n). \quad (3.14)$$

If we replace f_3^L by f , λ by -1 in (2.29), then (2.29) becomes (3.14). Thus by Lemma 2.2 (3), we have $f(n) = \frac{1}{2}(n^2 - n)f(2) - (n^2 - 2n)f(1)$.

If $\lambda \neq -1$, noticing $\bar{f}(n) = bn$ in (3.12), we have

$$(m - \lambda n)f(m) - (n - \lambda m)f(n) = (m - n)(f(m + n) - b^2nm). \quad (3.15)$$

Taking $n = 1, 2$ in (3.15) respectively, we have

$$(m - \lambda)f(m) - (1 - \lambda m)f(1) = (m - 1)(f(m + 1) - b^2m), \quad (3.16)$$

$$(m - 2\lambda)f(m) - (2 - \lambda m)f(2) = (m - 2)(f(m + 2) - 2b^2m). \quad (3.17)$$

Setting $n = 1$ and replacing m by $m + 1$ in (3.15), one has

$$(m + 1 - \lambda)f(m + 1) - (1 - \lambda(m + 1))f(1) = m(f(m + 2) - b^2(m + 1)). \quad (3.18)$$

Thus using (3.16)–(3.18), one can deduce

$$\begin{aligned} & (\lambda - 1)((\lambda + 2)m - 2\lambda)f(m) - b^2(\lambda - 2)m(m - 2)(m - 1) \\ &= m(m - 1)(\lambda m - 2)f(2) + m(m - 2)(2 - \lambda + \lambda^2 - 2\lambda m)f(1). \end{aligned} \quad (3.19)$$

Case 1 $\lambda = -2$.

In this case (3.19) gives

$$f(m) = \frac{1}{6}(m^3 - m)f(2) - \frac{1}{3}(m^3 - 4m)f(1) + \frac{b^2}{3}m(m - 2)(m - 1), \quad \forall m \in \mathbb{Z}.$$

Case 2 $\lambda = 1$.

In this case (3.15) gives

$$(m - n)(f(m) + f(n) - f(m + n) + b^2nm) = 0. \quad (3.20)$$

Obviously, $f(0) = 0$. Replacing m by $-n$ in (3.20), one can deduce

$$f(-n) + f(n) = b^2n^2, \quad \forall n \in \mathbb{Z}. \quad (3.21)$$

By (3.20), we have

$$f(m + n) = f(m) + f(n) + b^2nm \quad \text{for } m \neq n. \quad (3.22)$$

Combining (3.21), (3.22), we obtain

$$\begin{aligned} f(2m) &= f(3m) + f(-m) - 3b^2m^2 \\ &= f(2m) + f(m) + 2b^2m^2 + f(-2m) + f(m) - 2b^2m^2 - 3b^2m^2 \\ &= 2f(m) + b^2m^2. \end{aligned}$$

Thus (3.22) holds for all $m, n \in \mathbb{Z}$. Taking $m = 1$ and replacing n by $n - 1$ in (3.22), we obtain $f(n) = f(n - 1) + f(1) + b^2(n - 1)$. Hence, using induction on n , one can deduce $f(n) = nf(1) + \frac{1}{2}b^2n(n - 1)$.

Case 3 $\lambda \notin \{-2, 0, 1\}$.

Taking $m = 0$ in (3.15), one has $\lambda n f(0) = 0$ for all $n \in \mathbb{Z}$. Thus $f(0) = 0$. Using this, then setting $m = -1$ and $n = 1$ in (3.15), one can deduce

$$(1 + \lambda)f(-1) = 2b^2 - (1 + \lambda)f(1). \quad (3.23)$$

Then taking $n = 1$ in (3.15), we have

$$(m - 1)f(m + 1) = (m - \lambda)f(m) - (1 - \lambda m)f(1) + (m - 1)b^2m. \quad (3.24)$$

Setting $n = -1$ and replacing m by $m + 1$ in (3.15), we obtain

$$(m + 1 + \lambda)f(m + 1) + (1 + \lambda(m + 1))f(-1) = (m + 2)(f(m) + b^2(m + 1)). \quad (3.25)$$

Multiplying (3.25) by $(1 + \lambda)(m - 1)$, then using (3.23) and (3.24), one can deduce

$$\begin{aligned} & (1 - \lambda)(1 + \lambda)(2 + \lambda)f(m) \\ &= (1 - \lambda)(1 + \lambda)(2 + \lambda)m f(1) + b^2(1 - \lambda)(2 + \lambda)m(m - 1), \end{aligned}$$

since $\lambda \notin \{-2, \pm 1\}$, which gives $f(m) = m f(1) + \frac{b^2}{\lambda + 1}m(m - 1)$ for all $m \in \mathbb{Z}$.

Thus denoted by $f(2) = c$ and $f(1) = \bar{c}$, the lemma follows. \square

Lemma 3.3 (i) Let $\epsilon \in \{\pm 1\}$. If $\mu = 0$, then the map

$$\varphi_\epsilon : L_n \mapsto \epsilon L_{\epsilon n}, \quad Y_n \mapsto Y_{\epsilon n}, \quad M_n \mapsto \epsilon M_{\epsilon n},$$

is an automorphism of \mathcal{L} . The set $\{\varphi_\epsilon \mid \epsilon \in \{\pm 1\}\} \cong \mathbb{Z}_2$ forms a subgroup of $\text{Aut}\mathcal{L}$, where $\varphi_\epsilon \varphi_{\epsilon'} = \varphi_{\epsilon\epsilon'}$ for $\epsilon, \epsilon' \in \{\pm 1\}$.

(ii) For any $\alpha, \beta \in \mathbb{C}^*$, the map

$$\varphi_{\alpha, \beta} : L_n \mapsto \alpha^n L_n, \quad Y_n \mapsto \alpha^n \beta Y_n, \quad M_n \mapsto \alpha^n \beta^2 M_n$$

is an automorphism of \mathcal{L} . The set $\{\varphi_{\alpha, \beta} \mid \alpha, \beta \in \mathbb{C}^*\} \cong \mathbb{C}^* \times \mathbb{C}^*$ forms a subgroup of $\text{Aut}\mathcal{L}$, where $\varphi_{\alpha, \beta} \varphi_{\alpha', \beta'} = \varphi_{\alpha\alpha', \beta\beta'}$ for $\alpha, \alpha', \beta, \beta' \in \mathbb{C}^*$.

(iii) For any $b \in \mathbb{C}$, if $\mu = 0$, then the map

$$\phi_b(L_n) = \begin{cases} L_n + bn^3 M_n, & \lambda = -2, \\ L_n + bn^2 M_n, & \lambda = -1, \end{cases} \quad \phi_b(X_n) = X_n \text{ for } X \in \{Y, M\},$$

is an automorphism of \mathcal{L} . The set $\{\phi_b \mid b \in \mathbb{C}\} \cong \mathbb{C}$ forms a subgroup of $\text{Aut}\mathcal{L}$, where $\phi_b \phi_{b'} = \phi_{b+b'}$ for $b, b' \in \mathbb{C}$.

(iv) For any $e \in \mathbb{C}$, if $\mu = 0$, then the map ψ_e defined by

$$\begin{aligned} L_n &\mapsto L_n, & Y_n &\mapsto Y_n + enM_n, & M_n &\mapsto M_n, & \lambda &= -1, \\ L_n &\mapsto L_n, & Y_n &\mapsto Y_n + eM_n, & M_n &\mapsto M_n, & \lambda &= 1, \end{aligned}$$

is an automorphism of \mathcal{L} . The set $\{\psi_e | e \in \mathbb{C}\} \cong \mathbb{C}$ forms a subgroup of $\text{Aut}\mathcal{L}$, where $\psi_e\psi_{e'} = \psi_{e+e'}$ for $e, e' \in \mathbb{C}$.

Proof This follows from straightforward verifications, we omit the details here. \square

Introduce the following notation

$$\text{Inn}\mathcal{L} = \text{Span}\{\exp(a\text{ad}L_0 + \sum b_i\text{ad}Y_i + \sum c_j\text{ad}M_j) | a, b_i, c_j \in \mathbb{C}, i, j \in \mathbb{Z}\} \quad (3.26)$$

with $a = 0$ if $\mu \notin \mathbb{Z}$.

Theorem 3.4 (1) If $\mu \notin \frac{1}{2}\mathbb{Z}$, then $\text{Aut}\mathcal{L} \cong \text{Inn}\mathcal{L} \times \mathbb{C}^* \times \mathbb{C}^*$.

(2) If $\mu = 0$, then

$$\text{Aut}\mathcal{L} \cong \begin{cases} \text{Inn}\mathcal{L} \times \mathbb{C}^* \times \mathbb{Z}_2 \times \mathbb{C} & \text{if } \lambda = -2 \text{ or } 1, \\ \text{Inn}\mathcal{L} \times \mathbb{C}^* \times \mathbb{Z}_2 \times \mathbb{C} \times \mathbb{C} & \text{if } \lambda = -1, \\ \text{Inn}\mathcal{L} \times \mathbb{C}^* \times \mathbb{Z}_2 & \text{if } \lambda \notin \{-2, 0, \pm 1\}. \end{cases}$$

Proof Let σ be an automorphism of \mathcal{L} . By Lemma 3.1, one can write

$$\sigma(L_0) = \epsilon L_0 + b_0 Y_0 + c_0 M_0 + \sum_{0 \neq i \in S} b_i Y_i + \sum_{0 \neq j \in S'} c_j M_j, \quad (3.27)$$

for some $S, S' \subset \mathbb{Z}$. Construct an inner automorphism θ of \mathcal{L}

$$\theta = \exp\left(\sum_{0 \neq i \in S} \frac{ib_0 b_i}{(i+\mu)(i+2\mu)} \text{ad}M_i - \frac{b_i}{\epsilon(i+\mu)} \text{ad}Y_i\right) \exp\left(\sum_{0 \neq j \in S'} \frac{-c_j}{\epsilon(j+2\mu)} \text{ad}M_j\right). \quad (3.28)$$

One can check that $\theta^{-1}\sigma(L_0) = \epsilon L_0 + b_0 Y_0 + c_0 M_0$. Furthermore, since L_0 is a semisimple element of \mathcal{L} , then $\theta^{-1}\sigma(L_0)$ is also semisimple. If we denote A the matrix of $\text{ad}(\theta^{-1}\sigma(L_0))$ with respect to the basis $\{L_n, Y_n, M_n\}$, then

$$A = \begin{pmatrix} \epsilon n & b_0(\frac{\lambda+1}{2}n - \mu) & c_0(\lambda n - 2\mu) \\ 0 & \epsilon(n + \mu) & b_0 n \\ 0 & 0 & \epsilon(n + 2\mu) \end{pmatrix}.$$

On the other hand, it follows from the definition of *semisimple* that A can be diagonalized. Thus it is necessary that b_0 and $c_0(\lambda n - 2\mu)$ are equal to 0.

By Lemma 3.1, one can write

$$\begin{cases} \theta^{-1}\sigma(L_n) = \epsilon\bar{\alpha}^n L_{\epsilon n} + \sum_{i' \in \bar{S}} \bar{b}_{n_{i'}} Y_{i'} + \sum_{j' \in \bar{S}'} \bar{c}_{n_{j'}} M_{j'}, \\ \theta^{-1}\sigma(Y_n) = \bar{\alpha}^n \bar{\beta} Y_{\epsilon n} + \sum_{k \in \bar{S}''} \bar{e}_{n_k} M_k, \\ \theta^{-1}\sigma(M_n) = \epsilon\bar{\alpha}^n \bar{\beta}^2 M_{\epsilon n}, \end{cases} \quad (3.29)$$

for some $\bar{\alpha}, \bar{\beta} \in \mathbb{C}^*$, $\bar{b}_{n_{i'}}$, $\bar{c}_{n_{j'}}$ and $\bar{e}_{n_k} \in \mathbb{C}$, $\epsilon \in \{\pm 1\}$ with $\epsilon = 1$ for $\mu \notin \frac{1}{2}\mathbb{Z}$, $\bar{S}, \bar{S}', \bar{S}'' \subseteq \mathbb{Z}$.

Applying $\theta^{-1}\sigma$ to $(n + \mu)Y_n = [L_0, Y_n]$, we obtain

$$\begin{aligned} & (n + \mu) \left(\bar{\alpha}^n \bar{\beta} Y_{\epsilon n} + \sum_{k \in \bar{S}''} \bar{e}_{n_k} M_k \right) \\ &= [\epsilon L_0 + c_0 M_0, \bar{\alpha}^n \bar{\beta} Y_{\epsilon n}] + \sum_{k \in \bar{S}''} \bar{e}_{n_k} M_k = (n + \mu) \bar{\alpha}^n \bar{\beta} Y_{\epsilon n} + \sum_{k \in \bar{S}''} \epsilon \bar{e}_{n_k} (k + 2\mu) M_k, \end{aligned}$$

which gives

$$(n + \mu) \sum_{k \in \bar{S}''} \bar{e}_{n_k} M_k = \sum_{k \in \bar{S}''} \epsilon \bar{e}_{n_k} (k + 2\mu) M_k.$$

Comparing the coefficients of M_k , one has

$$\bar{e}_{n_k} (k + 2\mu - \epsilon(n + \mu)) = 0, \quad \forall k \in \bar{S}'' \quad (3.30)$$

Applying $\theta^{-1}\sigma$ to $nL_n = [L_0, L_n]$, we obtain

$$\begin{aligned} & n \left(\epsilon \bar{\alpha}^n L_{\epsilon n} + \sum_{i' \in \bar{S}} \bar{b}_{n_{i'}} Y_{i'} + \sum_{j' \in \bar{S}'} \bar{c}_{n_{j'}} M_{j'} \right) \\ &= [\epsilon L_0 + c_0 M_0, \epsilon \bar{\alpha}^n L_{\epsilon n} + \sum_{i' \in \bar{S}} \bar{b}_{n_{i'}} Y_{i'} + \sum_{j' \in \bar{S}'} \bar{c}_{n_{j'}} M_{j'}] \\ &= \epsilon n \bar{\alpha}^n L_{\epsilon n} + \sum_{i' \in \bar{S}} \epsilon \bar{b}_{n_{i'}} (i' + \mu) Y_{i'} + \sum_{j' \in \bar{S}'} \epsilon \bar{c}_{n_{j'}} (j' + 2\mu) M_{j'} + \epsilon \bar{\alpha}^n c_0 (\lambda \epsilon n - 2\mu) M_{\epsilon n}. \end{aligned} \quad (3.31)$$

Using $c_0(\lambda n - 2\mu) = 0$ and $\epsilon = 1$ for $\mu \notin \frac{1}{2}\mathbb{Z}$, one can deduce $\epsilon \bar{\alpha}^n c_0 (\lambda \epsilon n - 2\mu) = 0$. Using this in (3.31), one has

$$n \left(\sum_{i' \in \bar{S}} \bar{b}_{n_{i'}} Y_{i'} + \sum_{j' \in \bar{S}'} \bar{c}_{n_{j'}} M_{j'} \right) = \sum_{i' \in \bar{S}} \epsilon \bar{b}_{n_{i'}} (i' + \mu) Y_{i'} + \sum_{j' \in \bar{S}'} \epsilon \bar{c}_{n_{j'}} (j' + 2\mu) M_{j'}.$$

Comparing the coefficients of $Y_{i'}$ and $M_{j'}$ respectively, we have

$$\bar{b}_{n_{i'}} (i' - \epsilon n + \mu) = 0, \quad \forall i' \in \bar{S}, \quad \bar{c}_{n_{j'}} (j' - \epsilon n + 2\mu) = 0, \quad \forall j' \in \bar{S}'. \quad (3.32)$$

Case 1 $\mu \notin \frac{1}{2}\mathbb{Z}$.

Since $\mu \notin \frac{1}{2}\mathbb{Z}$, we have $\epsilon = 1$. Thus, by (3.30) and (3.32), we obtain $\bar{b}_{n_{i'}}, \bar{c}_{n_{j'}}$ and \bar{e}_{n_k} , are all equal to zero for all $i' \in \bar{S}, j' \in \bar{S}'$ and $k \in \bar{S}''$. Using this in (3.29), we obtain

$$\theta^{-1}\sigma(L_n) = \bar{\alpha}^n L_n, \quad \theta^{-1}\sigma(Y_n) = \bar{\alpha}^n \bar{\beta} Y_n, \quad \theta^{-1}\sigma(M_n) = \bar{\alpha}^n \bar{\beta}^2 M_n.$$

Let $\varphi_{\bar{\alpha}, \bar{\beta}}$ be the automorphism of \mathcal{L} as that given in Lemma 3.3 (ii). Then $\sigma = \theta \varphi_{\bar{\alpha}, \bar{\beta}}$.

Case 2 $\mu = 0$.

Since $\mu = 0$, we know that i', j' and k are all equal to ϵn for all $i' \in \bar{S}, j' \in \bar{S}'$ and $k \in \bar{S}''$ in (3.30) and (3.32). Using this in (3.29), we obtain

$$\begin{cases} \theta^{-1}\sigma(L_n) = \epsilon \bar{\alpha}^n L_{\epsilon n} + \bar{b}_n Y_{\epsilon n} + \bar{c}_n M_{\epsilon n}, \\ \theta^{-1}\sigma(Y_n) = \bar{\alpha}^n \bar{\beta} Y_{\epsilon n} + \bar{e}_n M_{\epsilon n}, \\ \theta^{-1}\sigma(M_n) = \epsilon \bar{\alpha}^n \bar{\beta}^2 M_{\epsilon n}, \end{cases} \quad (3.33)$$

for some \bar{b}_n, \bar{c}_n and $\bar{e}_n \in \mathbb{C}$. Let $\bar{\theta}$ be the automorphism of \mathcal{L} defined by

$$\bar{\theta}(X_n) = \bar{\alpha}^n X_n \quad \text{for } X \in \{L, Y, W\}.$$

Then $\bar{\theta}$ is an inner one. Set

$$\bar{f}(n) = \bar{b}_n (\bar{\alpha}^n \bar{\beta})^{-1}, \quad f(n) = \bar{c}_n (\epsilon \bar{\alpha}^n \bar{\beta}^2)^{-1}, \quad g(n) = \bar{e}_n (\epsilon \bar{\alpha}^n \bar{\beta}^2)^{-1}.$$

Define φ_ϵ and $\varphi_{1, \bar{\beta}}$ as those given in Lemma 3.3 (i) and (ii) respectively. Then we can write (3.33) as follows

$$\phi(L_n) = L_n + \bar{f}(n) Y_n + f(n) M_n, \quad \phi(Y_n) = Y_n + g(n) M_n, \quad \phi(M_n) = M_n,$$

where $\phi = (\varphi_\epsilon)^{-1} (\varphi_{1, \bar{\beta}})^{-1} (\bar{\theta})^{-1} \theta^{-1} \sigma$.

Subcase 1 $\lambda = -2$.

By Lemma 3.2, there exist some c_1, \bar{c}_1 and $b_1 \in \mathbb{C}$ such that

$$\begin{aligned} \phi(L_n) &= L_n + b_1 n Y_n + \left(\frac{c_1}{6} (n^3 - n) - \frac{\bar{c}_1}{3} (n^3 - 4n) + \frac{b_1^2}{3} n(n-2)(n-1) \right) M_n, \\ \phi(Y_n) &= Y_n - 2b_1 n M_n, \quad \phi(M_n) = M_n. \end{aligned}$$

Set $\theta_1 = \exp(-2b_1 \text{ad} Y_0 + \frac{1}{12}(c_1 - 8\bar{c}_1 - 4b_1^2) \text{ad} M_0)$. Then

$$(\theta_1)^{-1} \phi(L_n) = L_n + \frac{c_1 - 2\bar{c}_1 + 2b_1^2}{6} n^3 M_n, \quad (\theta_1)^{-1} \phi(Y_n) = Y_n, \quad (\theta_1)^{-1} \phi(M_n) = M_n.$$

Set $\alpha_1 = \frac{1}{6}(c_1 - 2\bar{c}_1 + 2b_1^2)$ and define ϕ_{α_1} as in Lemma 3.3 (iii). Then $(\phi_{\alpha_1})^{-1} (\theta_1)^{-1} \phi = Id$.

Subcase 2 $\lambda = -1$.

By Lemma 3.2, there exist some c_2, \bar{c}_2 and $b_2 \in \mathbb{C}$ such that

$$\phi(L_n) = L_n + \left(\frac{c_2}{2}(n^2 - n) - \bar{c}_2(n^2 - 2n)\right)M_n, \quad \phi(Y_n) = Y_n + b_2 n M_n, \quad \phi(M_n) = M_n.$$

Define ψ_{b_2} that given as in Lemma 3.3 (iv). Then

$$\begin{aligned} (\psi_{b_2})^{-1}\phi(L_n) &= L_n + \left(\frac{c_2}{2}(n^2 - n) - \bar{c}_2(n^2 - 2n)\right)M_n, \\ (\psi_{b_2})^{-1}\phi(Y_n) &= Y_n, \quad (\psi_{b_2})^{-1}\phi(M_n) = M_n. \end{aligned}$$

Set $\theta_2 = \exp\left(\frac{1}{2}(c_2 - 4\bar{c}_2)\text{ad}M_0\right)$. Then

$$\begin{aligned} (\theta_2)^{-1}(\psi_{b_2})^{-1}\phi(L_n) &= L_n + \frac{c_2 - 2\bar{c}_2}{2}n^2 M_n, \\ (\theta_2)^{-1}(\psi_{b_2})^{-1}\phi(Y_n) &= Y_n, \quad (\theta_2)^{-1}(\psi_{b_2})^{-1}\phi(M_n) = M_n. \end{aligned}$$

Set $\alpha_2 = \frac{1}{2}(c_2 - 2\bar{c}_2)$, let ϕ_{α_2} be as that given in Lemma 3.3(iii). Then $(\phi_{\alpha_2})^{-1}(\theta_2)^{-1}(\psi_{b_2})^{-1}\phi = Id$.

Subcase 3 $\lambda = 1$.

By Lemma 3.2, there exist some e, c_4, \bar{c}_4 and $b_4 \in \mathbb{C}$ such that

$$\begin{aligned} \phi(L_n) &= L_n + b_4 n Y_n + \left(c_4 n + \frac{b_4^2}{2}n(n-1)\right)M_n, \\ \phi(Y_n) &= Y_n + (b_4 n + e)M_n, \quad \phi(M_n) = M_n. \end{aligned}$$

Set $\theta_4 = \exp\left(b_4 \text{ad}Y_0 - \frac{1}{2}(b_4^2 - 2c_4)\text{ad}M_0\right)$. Then

$$(\theta_4)^{-1}\phi(L_n) = L_n, \quad (\theta_4)^{-1}\phi(Y_n) = Y_n + eM_n, \quad (\theta_4)^{-1}\phi(M_n) = M_n.$$

Let ψ_e be as that given in Lemma 3.3 (iv). Then $(\psi_e)^{-1}(\theta_4)^{-1}\phi = Id$.

Subcase 4 $\lambda \notin \{-2, 0, \pm 1\}$.

By Lemma 3.2, there exist some c_5, \bar{c}_5 and $b_5 \in \mathbb{C}$ such that

$$\begin{aligned} \phi(L_n) &= L_n + b_5 n Y_n + \left(\bar{c}_5 n + \frac{b_5^2}{\lambda + 1}n(n-1)\right)M_n, \\ \phi(Y_n) &= Y_n + \frac{2b_5}{\lambda + 1}nM_n, \quad \phi(M_n) = M_n. \end{aligned}$$

Set $\theta_5 = \exp\left(2b_5(\lambda + 1)^{-1}\text{ad}Y_0 + (\bar{c}_5(\lambda + 1) - b_5^2)(\lambda^2 + \lambda)^{-1}\text{ad}M_0\right)$. Then $(\theta_5)^{-1}\phi = Id$.

Recall that $\phi = (\varphi_\epsilon)^{-1}(\varphi_{1,\bar{\beta}})^{-1}(\bar{\theta})^{-1}\theta^{-1}\sigma$. Since $\text{Inn}\mathcal{L}$ is the normal subgroup of $\text{Aut}\mathcal{L}$, then there exist $\bar{\theta}_i \in \text{Inn}\mathcal{L}$ ($i = 1, 2, \dots, 5$) such that

$$\sigma = \begin{cases} \bar{\theta}_1\varphi_{1,\bar{\beta}}\varphi_\epsilon\phi_{\alpha_1} & \text{if } \lambda = -2, \\ \bar{\theta}_2\varphi_{1,\bar{\beta}}\varphi_\epsilon\phi_{\alpha_2}\psi_{b_2} & \text{if } \lambda = -1, \\ \bar{\theta}_4\varphi_{1,\bar{\beta}}\varphi_\epsilon\psi_e & \text{if } \lambda = 1, \\ \bar{\theta}_5\varphi_{1,\bar{\beta}}\varphi_\epsilon & \text{if } \lambda \notin \{-2, 0, \pm 1\}. \end{cases}$$

Obviously, $\text{Inn}\mathcal{L}$ satisfies (3.26). Hence the theorem follows from Lemma 3.2. \square

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